

**The high spin expansion of twist sector dimensions:  
the planar  $\mathcal{N} = 4$  super Yang-Mills theory**

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**Abstract**

This review is devoted to collecting some results on the high spin expansion of (minimal) anomalous dimension. Thanks to the recent rationale on integrability, planar  $\mathcal{N} = 4$  super Yang-Mills theory (or its  $\text{AdS}_5 \times \text{S}^5$  string counterpart) represents a very practicable field. Here the attention will be restricted to its sector of twist operators, although the analysis tools are quite general (in integrable theories). Some structures and ideas turn out to be general also for other sectors or gauge theories.

*Keywords:* Integrability; counting function; non-linear integral equation; AdS/CFT correspondence.

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# 1 Framework and beyond

We will move our investigation in within the maximally supersymmetric gauge theory in planar limit, *i.e.* for number of colours  $N \rightarrow \infty$  and coupling  $g_{YM} \rightarrow 0$ , so that the 't Hooft coupling

$$\lambda = g_{YM}^2 N = 8\pi^2 g^2 \quad (1.1)$$

may stay finite. Among the different sectors (perturbatively closed under renormalisation), we also pick up the twist  $sl(2)$  sector, spanned by local composite operators of trace form

$$\text{Tr}(\mathcal{D}^s \mathcal{Z}^L) + \dots, \quad (1.2)$$

where  $\mathcal{D}$  is the (light-cone) covariant derivative acting in all the possible ways on the  $L$  complex bosonic fields  $\mathcal{Z}$ . Trace ensures, of course, gauge invariance. The Lorentz spin of these operators is  $s$  and  $L$  is the  $R$ -charge which also coincides with the twist (classical dimension minus the spin). Besides, this sector may be described – thanks to the AdS/CFT correspondence [1] – by spinning folded closed strings on  $\text{AdS}_5 \times S^5$  spacetime with  $\text{AdS}_5$  and  $S^5$  angular momenta  $s$  and  $L$ , respectively [2, 3].

As being in a conformal model, suitable superpositions of operators form dilatation operator eigenvectors with definite dimensions (eigenvalues), which are made up of a classical part plus an anomalous one. For instance, in the sector (1.2) this spectral problem shows up dimensions

$$\Delta(g, s, L) = L + s + \gamma(g, s, L), \quad (1.3)$$

where  $\gamma(g, s, L)$  is the anomalous part. According to the AdS/CFT strong/weak coupling duality, the set of anomalous dimensions of composite operators in  $\mathcal{N} = 4$  SYM coincides with the energy spectrum of the  $\text{AdS}_5 \times S^5$  string theory ([1, 2, 3] and references therein), although the perturbative regimes are interchanged. The highly nontrivial problem of evaluating the anomalous part in  $\mathcal{N} = 4$  SYM was greatly simplified by the discovery of integrability in the purely bosonic  $so(6)$  sector at one loop [4]. Later on, this fact has been extended to all the gauge theory sectors and at all loops in a way which shows up integrability in a weaker sense, but still furnishes the investigators many powerful tools [5]. More in detail, any operator (*e.g.* of the form (1.2)) has been thought of as a state of a 'spin chain', whose hamiltonian is, of course, the dilatation operator itself, although the latter does not have an explicit expression of the spin chain form, but for the first few loops. Nevertheless, the large size (*asymptotic*) spectrum has turned out to be exactly described by certain Asymptotic Bethe Ansatz-like equations (the so-called Beisert-Staudacher equations, cf. [5, 6] and references therein). In other words, the anomalous dimensions coincide with the energies given by the Bethe Ansatz solutions (or roots): this is, of course, a great simplification of the initial spectral problem.

Unfortunately, this works only for infinitely long operators: anomalous dimensions of operators with finite length depend not only on Asymptotic Bethe Ansatz (ABA) data but also on finite size 'wrapping' corrections: in the perturbative expansion wrapping effects are observed [7] starting from the order  $g^{2L}$ . Recent progress has shown that a set of Thermodynamic Bethe Ansatz (TBA) equations [8] or, together with certain additional information, an equivalent  $Y$ -system of functional equations [9] provides a basis for exact (any length at any coupling) predictions for anomalous dimensions of planar  $\mathcal{N} = 4$  SYM.

However, in the  $sl(2)$  sector of  $\mathcal{N} = 4$  SYM relevance of wrapping effects seems to be reduced, even for short operators, if one goes to the high spin limit. For instance, findings of [10] have showed that, at least at twist two and up to five loops, wrapping corrections start contributing at order  $O((\ln s)^2/s^2)$ . This fact has pushed the idea of applying ABA techniques to the study of the high spin limit of twist operators, which - on the other hand - had already received much attention in the past literature. Indeed, the high spin behaviour of the anomalous dimension

$$\gamma(g, s, L) = \Delta(g, s, L) - L - s, \quad (1.4)$$

shows a Sudakov behaviour

$$\gamma(g, s, L) = f(g) \ln s + \dots, \quad (1.5)$$

determined by the so-called *universal* (since it does not depend on  $L$  or the flavour) *scaling function*,  $f(g)$  [11, 12, 13, 6, 14]. Actually, this behaviour is more general than in planar  $\mathcal{N} = 4$  SYM and the adjective *scaling* is due to the linear value of the *cusp anomalous dimension* with coefficient  $f(g)/2$  while the cusp angle tends to infinity [15]<sup>1</sup>. This large angle behaviour is due to the dominance by the lowest twist ( $= 2$ ) in the renormalisation of the vacuum expectation value of a cusped Wilson loop with a very large angle. In the end, for an infinite angle cusp (*i.e.* with one light-cone segment)  $f(g)$  equals twice the cusp anomalous dimension of a light-cone Wilson loop [17]. Additionally, the one-loop problem (and thus  $f(g)$ ) stays exactly the same for twist operators in QCD as long as the partonic helicities are aligned [18, 19]; this fact also has justified partially the great interest on the twist operators (1.2).

In general, the high spin limit of anomalous dimensions of twist  $L$  operators goes on as a series of logarithmic (inverse) powers<sup>2</sup>,

$$\gamma(g, s, L) = f(g) \ln s + f_{sl}(g, L) + \sum_{n=1}^{\infty} \gamma^{(n)}(g, L) (\ln s)^{-n} + O((\ln s)^{-\infty}), \quad (1.6)$$

*i.e.* looks, at this order, like an expansion in the large 'size'  $\ln s$ . Recently [6], the leading term (*i.e.*  $f(g)$ ) was obtained - in the hypothesis of being wrapping free - from the solution of a linear integral equation directly derived from the ABA via the root density approach. Moreover,  $f(g)$  was carefully studied and tested both in the weak [12, 6] and strong coupling limit [20, 21, 22, 23].

The sub-leading (constant) contribution  $f_{sl}(g, L)$  received also much attention. In the ABA framework, it was shown [24] to come from the solution of a non-linear integral equation (NLIE). Then, it was obtained [25] starting from a linear integral equation (LIE). Explicit weak and strong coupling expansions performed using the LIE of [25] are present in [26, 27] and agree with some string theory computations [28]<sup>3</sup>. Importantly, after results of [10] (*cf.* above) it may be inferred that both  $f(g)$  and  $f_{sl}(g, L)$  are exactly given by this approach based on the ABA (without wrapping corrections). Besides, both terms fix the  $1/s$  coefficients via the reciprocity relation which may still be, consistently with [10], wrapping free.

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<sup>1</sup>Polyakov noticed as first that for cusped Wilson loop vacuum expectation value the charge renormalisation is not enough as in the non-cusped case, because of an extra logarithmic divergence due to the high bremsstrahlung at the cusp. He was led to consider cusps because of their importance in the loop dynamics (in euclidian space-time) [16].

<sup>2</sup>With  $O((\ln s)^{-\infty})$  we indicate terms going to zero faster than any inverse powers of  $\ln s$ .

<sup>3</sup>Agreement between ABA [29] and string theory [28] results is now established up to two loops.

The latter reasoning would support that even sub-logarithmic terms from ABA be exact, but it is even more unclear if and at what extent this might be trusted. In [30] we studied and computed in a systematic way the ABA contribution to them, by using a set of integral equations. Possible wrapping corrections to our results are still to be determined. In this respect, in addition to TBA findings, also results on the string side of the correspondence in the spirit of [28] could be of fundamental importance. In fact, for instance, the one-loop contribution to the  $1/\ln s$  term in the second of [28] comes partially from the  $AdS_5$  modes (wrapping free) and partially from the  $S^5$  ones (wrapping)<sup>4</sup>. Nevertheless, a better understanding might come from exact (maybe TBA) calculations.

Wrapping effects should be negligible<sup>5</sup> if one takes the infinite twist limit

$$s \rightarrow \infty, \quad L \rightarrow \infty, \quad j = \frac{L-2}{\ln s} \quad \text{fixed}, \quad (1.7)$$

and restricts to the study of the scaling functions  $f_n(g)$ ,  $f_n^{(r)}(g)$  appearing in the expansion

$$\gamma(g, s, L) = \ln s \sum_{n=0}^{\infty} f_n(g) j^n + \sum_{r=0}^{\infty} (\ln s)^{-r} \sum_{n=0}^{\infty} f_n^{(r)}(g) j^n + O((\ln s)^{-\infty}). \quad (1.8)$$

For this reason, a reasonable amount of activity was also devoted to the study of limit (1.7), using ABA techniques. Results concerning  $f_n(g)$  at weak coupling are present in [24]. Strong coupling behaviour of  $f_n(g)$  was studied in [31, 32, 33, 34] by relying on linear integral equations. A detailed study of  $f_n^{(r)}(g)$ ,  $r \geq 0$ , can be found in [35, 30].

In this review we want to give a summary of our activity in the framework of high spin limit of twist operators. We will first give a general description of the method we have used and whose main tools are integral equations derived from the ABA equations. Then, we will briefly report the most important results we obtained.

This review is organised as follows. In Section 2 we review the ABA equations for twist operators. We illustrate the properties of the minimal anomalous dimension operators and explain a technique which allows exact computations of ABA contributions to the anomalous dimension by using the so-called Non-Linear Integral Equation (NLIE). In Section 3 we specialise to the high spin limit and show that, if one neglects  $O((\ln s)^{-\infty})$  terms, asymptotic anomalous dimension can be computed by relying on integral equations. In Section 4 we study the high spin limit at fixed twist. In Section 5 results in the scaling limit (1.7) are discussed.

## 2 All-loop ABA and the (N)LIE

Let us recall the ABA equations [5, 6] for the  $sl(2)$  sector,

$$\left( \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L \left( \frac{1 + \frac{g^2}{2x_k^+}}{1 + \frac{g^2}{2x_k^-}} \right)^L = \prod_{\substack{j=1 \\ j \neq k}}^s \frac{u_k - u_j - i}{u_k - u_j + i} \left( \frac{1 - \frac{g^2}{2x_k^+ x_j^-}}{1 - \frac{g^2}{2x_k^- x_j^+}} \right)^2 e^{2i\theta(u_k, u_j)}, \quad (2.1)$$

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<sup>4</sup>We wish to thank N. Gromov who pointed out this double origin as he sees it clearly from the algebraic curve approach which reproduces one-loop ABA plus some 'vacuum oscillations' (wrapping).

<sup>5</sup>This should be true at least for small values of  $g$ . But, again, this reasoning is not rigorous.

where

$$x_k^\pm = x^\pm(u_k) = x(u_k \pm i/2), \quad x(u) = \frac{u}{2} \left[ 1 + \sqrt{1 - \frac{2g^2}{u^2}} \right], \quad \lambda = 8\pi^2 g^2, \quad (2.2)$$

$\lambda$  being the 't Hooft coupling. The so-called dressing factor [36, 37, 6]  $\theta(u, v)$  is given by

$$\theta(u, v) = \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} \beta_{r,r+1+2\nu}(g) [q_r(u) q_{r+1+2\nu}(v) - q_r(v) q_{r+1+2\nu}(u)], \quad (2.3)$$

the functions  $\beta_{r,r+1+2\nu}(g) = g^{2r+2\nu-2} 2^{1-r-\nu} c_{r,r+1+2\nu}(g)$  being

$$\begin{aligned} \beta_{r,r+1+2\nu}(g) &= 2 \sum_{\mu=\nu}^{\infty} \frac{g^{2r+2\nu+2\mu}}{2^{r+\mu+\nu}} (-1)^{r+\mu+1} \frac{(r-1)(r+2\nu)}{2\mu+1} \cdot \\ &\quad \cdot \binom{2\mu+1}{\mu-r-\nu+1} \binom{2\mu+1}{\mu-\nu} \zeta(2\mu+1) \end{aligned} \quad (2.4)$$

and  $q_r(u)$  being the density of the  $r$ -th charge

$$q_r(u) = \frac{i}{r-1} \left[ \left( \frac{1}{x^+(u)} \right)^{r-1} - \left( \frac{1}{x^-(u)} \right)^{r-1} \right]. \quad (2.5)$$

It is now clear that configurations of Bethe roots, i.e. solutions of (2.1), and the corresponding eigenvalues of the energy are related respectively to composite operators and their anomalous dimensions in the  $sl(2)$  sector of  $\mathcal{N} = 4$  SYM.

In the  $sl(2)$  sector states of twist  $L$  are described by an even number  $s$  of real Bethe roots  $u_k$  which satisfy (2.1). Bethe roots localize in an interval  $[-b, b]$  of the real line. In addition to Bethe roots, also  $L$  real 'holes' [11, 38, 6, 24, 25] are present (a better explanation of the nature of holes will be given in the following). For any state, two holes reside outside the interval  $[-b, b]$  and the remaining  $L-2$  holes lie inside this interval. We will indicate with  $u_h^{(i)}$  these 'internal' holes.

In this paper we will focus on the minimal anomalous dimension state. For such a state the positions of both roots and holes are symmetric with respect to the origin. For what concerns the internal holes, they all concentrate near the origin, with no roots lying in between.

In general, an efficient way to treat states described by solutions of a non-linear set of Bethe Ansatz equations consists in writing a non-linear integral equation, which is completely equivalent to them. The non-linear integral equation is satisfied by the counting function  $Z(u)$ , which in the case (2.1) reads as

$$Z(u) = \Phi(u) - \sum_{k=1}^s \phi(u, u_k), \quad (2.6)$$

where

$$\Phi(u) = -2L \arctan 2u - iL \ln \left( \frac{1 + \frac{g^2}{2x^-(u)^2}}{1 + \frac{g^2}{2x^+(u)^2}} \right), \quad (2.7)$$

$$\phi(u, v) = 2 \arctan(u - v) - 2i \left[ \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u, v) \right]. \quad (2.8)$$

It follows from its definition that the counting function  $Z(u)$ , as a function of the real variable  $u$ , is a monotonously decreasing function and that

$$\lim_{u \rightarrow \pm\infty} Z(u) = \mp\pi(L+s). \quad (2.9)$$

Therefore, there are  $L+s$  real points  $v_k$  such that  $e^{iZ(v_k)} = (-1)^{L+1}$ . It is a simple consequence of the definition of  $Z(u)$  that  $s$  of them coincide with the Bethe roots. The remaining  $L$  points are called 'holes' and their role will be of fundamental importance in what follows. As we anticipated before, for the minimal anomalous dimension state the internal holes concentrate near the origin, i.e. their positions  $u_h^{(i)}$  are determined by the relations

$$Z(u_h^{(i)}) = \pi(2h+1-L), \quad h = 1, \dots, L-2. \quad (2.10)$$

Excited states are obtained by making different choices for the 'quantum numbers'  $h$ . It follows from the structure of (2.10) that, for any state,  $u_h^{(i)}$  depend in a non-linear way on the counting function  $Z(u)$ .

The non-linear integral equation for  $Z(u)$  is written by using a modification of the standard ideas underlying the procedure concerning the excited state NLIE [39], this modification being dictated by the physical situation with two 'important' external holes. Suppose that in the interval  $[-b, b]$  of the real line  $s$  Bethe roots and  $L-2$  holes are present. Then, by using Cauchy theorem we can express a sum over the Bethe roots of an observable  $O(u)$  as

$$\begin{aligned} \sum_{k=1}^s O(u_k) &= - \int_{-b}^b \frac{dv}{2\pi} O(v) Z'(v) + \text{Im} \int_{-b}^b \frac{dv}{\pi} O(v-i\epsilon) \frac{d}{dv} \ln[1 + (-1)^L e^{iZ(v-i\epsilon)}] + \\ &+ \text{Im} \int_0^{-\epsilon} \frac{dy}{\pi} O(-b+iy) \frac{d}{dy} \ln[1 + (-1)^L e^{iZ(-b+iy)}] + \\ &+ \text{Im} \int_{-\epsilon}^0 \frac{dy}{\pi} O(b+iy) \frac{d}{dy} \ln[1 + (-1)^L e^{iZ(b+iy)}] - \sum_{h=1}^{L-2} O(u_h^{(i)}). \end{aligned} \quad (2.11)$$

The right hand side of (2.11) does not depend on  $\epsilon > 0$  as far as no poles of the integrands  $O(w) \frac{d}{dw} \ln[1 + (-1)^L e^{iZ(w)}]$  lie in the region  $|\text{Im } w| \leq \epsilon$ ,  $|\text{Re } w| \leq b$ . In addition,  $\epsilon$  must be kept sufficiently small, in such a way that

$$|e^{iZ(z)}| < 1, \quad (2.12)$$

where  $z$  belongs to the integration contour of the last three integral terms in (2.11). This condition is assured by the monotonicity of the counting function, i.e. in our case  $Z'(v) < 0$ ,  $v \in [-b, b]$ . We apply (2.11) to the sum over the Bethe roots contained in (2.6):

$$\begin{aligned} Z(u) &= \Phi(u) + \int_{-b}^b \frac{dv}{2\pi} \phi(u, v) Z'(v) + \sum_{h=1}^{L-2} \phi(u, u_h^{(i)}) - \text{Im} \int_{-b}^b \frac{dv}{\pi} \phi(u, v-i\epsilon) \frac{d}{dv} \ln[1 + (-1)^L e^{iZ(v-i\epsilon)}] - \\ &- \text{Im} \int_0^{-\epsilon} \frac{dy}{\pi} \phi(u, -b+iy) \frac{d}{dy} \ln[1 + (-1)^L e^{iZ(-b+iy)}] - \text{Im} \int_{-\epsilon}^0 \frac{dy}{\pi} \phi(u, b+iy) \frac{d}{dy} \ln[1 + (-1)^L e^{iZ(b+iy)}]. \end{aligned} \quad (2.13)$$

What we have obtained is a non-linear integral equation - for the counting function  $Z(u)$  - which describes - in a way which is alternative to the Bethe Ansatz equations - the minimal anomalous dimension state. Since Bethe roots are localised in an interval of the real axis (i.e.  $b < +\infty$ ), NLIE (2.13) is different from the equation introduced in [39], which contains integrations over the entire real axis. This difference reveals crucial in the specific problem of twist operators in the  $sl(2)$  sector. Indeed, important simplifications to the structure of (2.13) arise in the high spin limit: if we decide to neglect terms going to zero faster than any inverse power of  $\ln s$ , then it is possible to get rid of the last three non-linear terms in (2.13). We are now going to show this important property.

Coming back to (2.11), we first use (2.12) in order to replace all the  $\ln[1 + (-1)^L e^{iZ(z)}]$  with  $\sum_{n=1}^{\infty} (-1)^{n+1} (-1)^{nL} \frac{e^{inZ(z)}}{n}$ . We obtain:

$$\begin{aligned} \sum_{k=1}^s O(u_k) &= - \int_{-b}^b \frac{dv}{2\pi} O(v) Z'(v) + \text{Im} \int_{-b}^b \frac{dv}{\pi} O(v - i\epsilon) \frac{d}{dv} \sum_{n=1}^{\infty} (-1)^{n+1} (-1)^{nL} \frac{e^{inZ(v-i\epsilon)}}{n} + \\ &+ \text{Im} \int_0^{-\epsilon} \frac{dy}{\pi} O(-b + iy) \frac{d}{dy} \sum_{n=1}^{\infty} (-1)^{n+1} (-1)^{nL} \frac{e^{inZ(-b+iy)}}{n} + \\ &+ \text{Im} \int_{-\epsilon}^0 \frac{dy}{\pi} O(b + iy) \frac{d}{dy} \sum_{n=1}^{\infty} (-1)^{n+1} (-1)^{nL} \frac{e^{inZ(b+iy)}}{n} - \sum_{h=1}^{L-2} O(u_h^{(i)}). \end{aligned} \quad (2.14)$$

In order to evaluate the non-linear terms in this expression,

$$\begin{aligned} NL &= \text{Im} \int_{-b}^b \frac{dv}{\pi} O(v - i\epsilon) \frac{d}{dv} \sum_{n=1}^{\infty} (-1)^{n+1} (-1)^{nL} \frac{e^{inZ(v-i\epsilon)}}{n} + \\ &+ \text{Im} \int_0^{-\epsilon} \frac{dy}{\pi} O(-b + iy) \frac{d}{dy} \sum_{n=1}^{\infty} (-1)^{n+1} (-1)^{nL} \frac{e^{inZ(-b+iy)}}{n} + \\ &+ \text{Im} \int_{-\epsilon}^0 \frac{dy}{\pi} O(b + iy) \frac{d}{dy} \sum_{n=1}^{\infty} (-1)^{n+1} (-1)^{nL} \frac{e^{inZ(b+iy)}}{n}, \end{aligned}$$

we first assume that it is possible to exchange the series with the integrals. Then, we use the following formulæ

$$\int^v dv O(v - i\epsilon) \frac{d}{dv} e^{inZ(v-i\epsilon)} = e^{inx} \sum_{k=0}^{\infty} \left( \frac{i}{n} \right)^k \frac{d^k}{dx^k} O[Z^{-1}(x)] \Big|_{x=Z(v-i\epsilon)}, \quad (2.15)$$

$$\int^y dy O(\pm b + iy) \frac{d}{dy} e^{inZ(\pm b+iy)} = e^{inx} \sum_{k=0}^{\infty} \left( \frac{i}{n} \right)^k \frac{d^k}{dx^k} O[Z^{-1}(x)] \Big|_{x=Z(\pm b+iy)}. \quad (2.16)$$

We need to remark that results (2.15, 2.16) are correct if the above infinite sums make sense either as convergent or asymptotic series. In our case, we can use (2.15, 2.16), since the series we will get are asymptotic.

Using (2.15, 2.16) in the evaluation of the above non-linear terms, the dependence on  $\epsilon$  cancels out, as it should be, and we are left with

$$NL = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^{nL}}{\pi n} \left[ \sum_{k=0}^{\infty} \frac{i^{2k}}{n^{2k}} \sin nx \frac{d^{2k}}{dx^{2k}} O(Z^{-1}(x)) + \sum_{k=0}^{\infty} \frac{i^{2k}}{n^{2k+1}} \cos nx \frac{d^{2k+1}}{dx^{2k+1}} O(Z^{-1}(x)) \right] \Bigg|_{x=Z(-b)}^{x=Z(b)}. \quad (2.17)$$

Now, we are allowed to choose  $b$  in such a way that  $e^{iZ(\pm b)} = (-1)^L$ : therefore, since  $O(v)$  is bounded, all the terms in (2.17) proportional to the sine function are zero. Thus, we are left only with terms containing the cosine function, i.e., after summing over  $n$ ,

$$NL = - \sum_{k=0}^{\infty} \frac{(2\pi)^{2k+1}}{(2k+2)!} B_{2k+2} \left( \frac{1}{2} \right) \left[ \frac{\partial}{\partial x^{2k+1}} O(Z^{-1}(x)) \right] \Bigg|_{x=Z(-b)}^{x=Z(b)}, \quad (2.18)$$

where  $B_k(x)$  is the Bernoulli polynomial. Relation (2.18) allows to write the final formula:

$$\sum_{k=1}^s O(u_k) = - \int_{-b}^b \frac{dv}{2\pi} O(v) Z'(v) - \sum_{h=1}^{L-2} O(u_h^{(i)}) - \sum_{k=0}^{\infty} \frac{(2\pi)^{2k+1}}{(2k+2)!} B_{2k+2} \left( \frac{1}{2} \right) \left[ \frac{\partial}{\partial x^{2k+1}} O(Z^{-1}(x)) \right] \Bigg|_{x=Z(-b)}^{x=Z(b)}. \quad (2.19)$$

The following expressions

$$\begin{aligned} \frac{d}{dx} O(Z^{-1}(x)) \Bigg|_{x=Z(-b)}^{x=Z(b)} &= 2 \frac{O'(b)}{Z'(b)}, \\ \frac{d^2}{dx^2} O(Z^{-1}(x)) \Bigg|_{x=Z(-b)}^{x=Z(b)} &= 2 \frac{O''(b) - O'(b) \frac{Z''(b)}{Z'(b)}}{[Z'(b)]^2}, \\ \frac{d^3}{dx^3} O(Z^{-1}(x)) \Bigg|_{x=Z(-b)}^{x=Z(b)} &= 2 \frac{O'''(b) - 3O''(b) \frac{Z''(b)}{Z'(b)} - O'(b) \frac{Z'''(b)}{Z'(b)} + 3O'(b) \frac{(Z''(b))^2}{(Z'(b))^2}}{[Z'(b)]^3} \end{aligned}$$

give an idea of the form of the first nonlinear terms appearing in (2.19). Next step is to apply (2.18) to the non-linear integral terms (the last three ones) contained in (2.13). We have to replace  $O(v)$  with  $-\phi(u, v)$ . From the form of (2.18) we realise that such non-linear terms are proportional to derivatives

$$\frac{d^n}{dv^n} \phi(u, v) \Bigg|_{\pm b}, \quad n \geq 1. \quad (2.20)$$

When  $s \rightarrow \infty$ ,  $b = \frac{s}{2} (1 + O(\frac{1}{s}))$  and for such derivatives

$$\frac{d^n}{dv^n} \phi(u, v) \Bigg|_{\pm b} = O\left(\frac{1}{b^{n+1}}\right). \quad (2.21)$$

On the other hand, for the derivatives of the counting function we can give the estimates

$$\frac{d^n}{dv^n} Z(v) \Bigg|_{\pm b} = O\left(\frac{1}{b^n}\right). \quad (2.22)$$

Putting all together, we conclude that when the spin  $s \rightarrow +\infty$  the non-linear integral terms contained in (2.13) are  $O(1/b) = O((\ln s)^{-\infty})$ . Therefore, if we decide to neglect terms  $O((\ln s)^{-\infty})$  and to focus only on corrections  $O((\ln s)^{-n})$ , we are entitled not to consider all the non-linear integral terms in (2.13). In this case we are left with

$$Z(u) = \Phi(u) + \int_{-b}^b \frac{dv}{2\pi} \phi(u, v) Z'(v) + \sum_{h=1}^{L-2} \phi(u, u_h^{(i)}) + O((\ln s)^{-\infty}) , \quad (2.23)$$

and this equation has to be solved together with condition (2.10), which fixes the holes positions  $u_h^{(i)}$  in terms of the counting function  $Z(u)$ . It is important to remark that within our approximations non-linearity with respect to  $Z(u)$  enters only through (2.10).

Equations (2.23, 2.10) are our starting point for the study of the high spin limit. They will be worked out in next section.

### 3 Integral equations for the logarithmic terms

Let us consider integral equation (2.23) satisfied by the counting function in the high spin limit. Passing to derivatives, we define

$$\sigma(u) = Z'(u) . \quad (3.1)$$

We have

$$\sigma(u) = \Phi'(u) + \int_{-b}^b \frac{dv}{2\pi} \frac{d}{du} \phi(u, v) \sigma(v) + \sum_{h=1}^{L-2} \frac{d}{du} \phi(u, u_h^{(i)}) + O((\ln s)^{-\infty}) . \quad (3.2)$$

We now decompose such equation in its one loop (with index 0) and higher than one loop (with index  $H$ ) contributions. We set  $\Phi(u) = \Phi_0(u) + \Phi_H(u)$ ,  $\phi(u, v) = \phi_0(u, v) + \phi_H(u, v)$ ,  $\sigma(u) = \sigma_0(u) + \sigma_H(u)$ , where

$$\Phi_0(u) = -2L \arctan 2u , \quad \Phi_H(u) = -iL \ln \left( \frac{1 + \frac{g^2}{2x^-(u)^2}}{1 + \frac{g^2}{2x^+(u)^2}} \right) , \quad (3.3)$$

$$\phi_0(u, v) = 2 \arctan(u - v) , \quad \phi_H(u, v) = -2i \left[ \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u, v) \right] , \quad (3.4)$$

and where, after neglecting quantities which are  $O((\ln s)^{-\infty})$ ,  $\sigma_0(u)$  and  $\sigma_H(u)$  satisfy

$$\sigma_0(u) = \Phi'_0(u) + \int_{-b_0}^{b_0} \frac{dv}{2\pi} \frac{d}{du} \phi_0(u, v) \sigma_0(v) + \sum_{h=1}^{L-2} \frac{d}{du} \phi_0(u, \bar{u}_h^{(i)}) + O((\ln s)^{-\infty}) , \quad (3.5)$$

$$\begin{aligned} \sigma_H(u) &= \Phi'_H(u) + \int_{-b}^b \frac{dv}{2\pi} \frac{d}{du} \phi_H(u, v) \sigma_H(v) + \int_{-b_0}^{b_0} \frac{dv}{2\pi} \frac{d}{du} \phi_H(u, v) \sigma_0(v) + \\ &+ \int_{-b}^b \frac{dv}{2\pi} \frac{d}{du} \phi_0(u, v) \sigma_H(v) + \sum_{h=1}^{L-2} \frac{d}{du} [\phi(u, u_h^{(i)}) - \phi_0(u, \bar{u}_h^{(i)})] + O((\ln s)^{-\infty}) . \end{aligned} \quad (3.6)$$

In (3.5, 3.6) the notation  $\bar{u}_h^{(i)}$  stands for the one loop component of the position of the  $h$ -th internal hole. In order to solve the one loop equation (3.5), we consider (see 3.52 of [25]) a function  $\sigma_0^{(s)}(u)$ , whose Fourier transform<sup>6</sup> reads

$$\hat{\sigma}_0^{(s)}(k) = -4\pi \frac{\frac{L}{2} - e^{-\frac{|k|}{2}} \cos(ks/\sqrt{2})}{2 \sinh \frac{|k|}{2}} + 2\pi \frac{e^{-\frac{|k|}{2}}}{2 \sinh \frac{|k|}{2}} \sum_{h=1}^{L-2} e^{ik\bar{u}_h^{(i)}} - (4\pi \ln 2) \delta(k). \quad (3.8)$$

Function (3.8) satisfies the following important property [25],

$$\int_{-b_0}^{b_0} du f(u) \sigma_0(u) = \int_{-\infty}^{+\infty} du f(u) \sigma_0^{(s)}(u) + O((\ln s)^{-\infty}), \quad (3.9)$$

which allows to extend to the whole real axis the integrations involving  $\sigma_0(u)$ : a look at many loops equation (3.6) shows that this is just the property we need in order to try to solve it.

It follows from results in [12] and from numerical checks that we can extend to the entire real axis the integrations involving  $\sigma_H(u)$ : what we are missing are  $O((\ln s)^{-\infty})$  terms. Putting all these pieces together, the equation satisfied by  $\sigma_H(u)$  is

$$\begin{aligned} \sigma_H(u) &= \Phi'_H(u) + \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \frac{d}{du} \phi_H(u, v) [\sigma_H(v) + \sigma_0^{(s)}(v)] + \\ &+ \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \frac{d}{du} \phi_0(u, v) \sigma_H(v) + \sum_{h=1}^{L-2} \frac{d}{du} [\phi(u, u_h^{(i)}) - \phi_0(u, \bar{u}_h^{(i)})] + O((\ln s)^{-\infty}). \end{aligned} \quad (3.10)$$

It is now convenient to pass to Fourier transforms. We have

$$\begin{aligned} \hat{\Phi}_H(k) &= \frac{2\pi L}{ik} e^{-\frac{|k|}{2}} [1 - J_0(\sqrt{2}gk)], \\ \hat{\phi}_H(k, t) &= -8i\pi^2 \frac{e^{-\frac{|t|+|k|}{2}}}{k|t|} \left[ \sum_{r=1}^{\infty} r(-1)^{r+1} J_r(\sqrt{2}gk) J_r(\sqrt{2}gt) \frac{1 - \text{sgn}(kt)}{2} + \right. \\ &+ \text{sgn}(t) \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} c_{r, r+1+2\nu}(g) (-1)^{r+\nu} \left( J_{r-1}(\sqrt{2}gk) J_{r+2\nu}(\sqrt{2}gt) - \right. \\ &\left. \left. - J_{r-1}(\sqrt{2}gt) J_{r+2\nu}(\sqrt{2}gk) \right) \right]. \end{aligned} \quad (3.12)$$

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<sup>6</sup>We adopt the following definition for the Fourier transform  $\hat{f}(k)$  of a function  $f(u)$ :

$$\hat{f}(k) = \int_{-\infty}^{+\infty} du e^{-iku} f(u). \quad (3.7)$$

Therefore, the Fourier transform of (3.10) reads:

$$\begin{aligned}
\hat{\sigma}_H(k) &= \frac{\pi L}{\sinh \frac{|k|}{2}} [1 - J_0(\sqrt{2}gk)] + \frac{1}{\sinh \frac{|k|}{2}} \int_{-\infty}^{+\infty} \frac{dt}{|t|} \left[ \sum_{r=1}^{\infty} r(-1)^{r+1} J_r(\sqrt{2}gk) J_r(\sqrt{2}gt) \frac{1 - \text{sgn}(kt)}{2} e^{-\frac{|t|}{2}} + \right. \\
&+ \text{sgn}(t) \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} c_{r,r+1+2\nu}(g) (-1)^{r+\nu} e^{-\frac{|t|}{2}} \left( J_{r-1}(\sqrt{2}gk) J_{r+2\nu}(\sqrt{2}gt) - J_{r-1}(\sqrt{2}gt) J_{r+2\nu}(\sqrt{2}gk) \right) \Big] \cdot \\
&\cdot \left[ \hat{\sigma}_H(t) + \hat{\sigma}_0^{(s)}(t) + 2\pi \sum_{h=1}^{L-2} e^{itu_h^{(i)}} \right] + 2\pi \frac{e^{-\frac{|k|}{2}}}{2 \sinh \frac{|k|}{2}} \sum_{h=1}^{L-2} \left[ e^{-iku_h^{(i)}} - e^{-ik\bar{u}_h^{(i)}} \right] + O((\ln s)^{-\infty}) . \quad (3.13)
\end{aligned}$$

Inserting in (3.13) the expression (3.8) for  $\hat{\sigma}_0^{(s)}(t)$ , introducing the 'magic' kernel  $\hat{K}(t, t')$ , defined in [6] as

$$\hat{K}(t, t') = \frac{2}{tt'} \left[ \sum_{n=1}^{\infty} n J_n(t) J_n(t') + 2 \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} (-1)^{k+l} c_{2k+1, 2l+2}(g) J_{2k}(t) J_{2l+1}(t') \right] , \quad (3.14)$$

and restricting to  $k \geq 0$ , we finally get the integral equation,

$$\begin{aligned}
\hat{\sigma}_H(k) &= \frac{\pi L}{\sinh \frac{k}{2}} [1 - J_0(\sqrt{2}gk)] - g^2 \frac{k}{\sinh \frac{k}{2}} \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \cdot \\
&\cdot \left\{ \hat{\sigma}_H(t) - 4\pi \frac{\frac{L}{2} - e^{-\frac{|t|}{2}} \cos(ts/\sqrt{2})}{2 \sinh \frac{|t|}{2}} - (4\pi \ln 2) \delta(t) + 2\pi \frac{e^{-\frac{|t|}{2}}}{2 \sinh \frac{|t|}{2}} \sum_{h=1}^{L-2} \cos t \bar{u}_h^{(i)} + 2\pi \sum_{h=1}^{L-2} \cos t u_h^{(i)} \right\} + \\
&+ \pi \frac{e^{-\frac{k}{2}}}{\sinh \frac{k}{2}} \sum_{h=1}^{L-2} \left[ \cos k u_h^{(i)} - \cos k \bar{u}_h^{(i)} \right] . \quad (3.15)
\end{aligned}$$

Since we are interested in the computation of the anomalous dimension, we go back to the general formula (2.19) for the evaluation of observables,

$$\sum_{k=1}^s O(u_k) = - \int_{-b}^b \frac{dv}{2\pi} O(v) Z'(v) - \sum_{h=1}^{L-2} O(u_h^{(i)}) - \sum_{k=0}^{\infty} \frac{(2\pi)^{2k+1}}{(2k+2)!} B_{2k+2} \left( \frac{1}{2} \right) \left[ \frac{\partial}{\partial x^{2k+1}} O(Z^{-1}(x)) \right] \Big|_{x=Z(-b)}^{x=Z(b)} . \quad (3.16)$$

If we specialise to the energy (anomalous dimension), we have

$$O(v) = e(v) = \frac{ig^2}{x^+(v)} - \frac{ig^2}{x^-(v)} . \quad (3.17)$$

Again, in the large spin limit, one has (with  $n \geq 1$ ):

$$\left. \frac{d^n}{dv^n} e(v) \right|_{\pm b} = O\left(\frac{1}{b^{n+1}}\right) , \quad (3.18)$$

together with the estimate (2.22) for the counting function. It follows that the non-linear terms contained in the expression (3.16) for the energy are  $O(1/b) = O(1/s)$ . Therefore, if we neglect  $O((\ln s)^{-\infty})$  terms, we are allowed to work with only the linear expression

$$\gamma(g, s, L) = \sum_{k=1}^s e(u_k) = - \int_{-b}^b \frac{dv}{2\pi} e(v) Z'(v) - \sum_{h=1}^{L-2} e(u_h^{(i)}) + O((\ln s)^{-\infty}) , \quad (3.19)$$

which we find convenient to write in terms of the one loop and higher than one loop densities:

$$\gamma(g, s, L) = \sum_{k=1}^s e(u_k) = - \int_{-b}^b \frac{dv}{2\pi} e(v) \sigma_H(v) - \int_{-b_0}^{b_0} \frac{dv}{2\pi} e(v) \sigma_0(v) - \sum_{h=1}^{L-2} e(u_h^{(i)}) + O((\ln s)^{-\infty}) . \quad (3.20)$$

Extending the domains of integration to the real axis does not give problems, since we are neglecting  $O((\ln s)^{-\infty})$  terms: we get

$$\gamma(g, s, L) = \sum_{k=1}^s e(u_k) = - \int_{-\infty}^{+\infty} \frac{dv}{2\pi} e(v) \sigma_H(v) - \int_{-\infty}^{+\infty} \frac{dv}{2\pi} e(v) \sigma_0^{(s)}(v) - \sum_{h=1}^{L-2} e(u_h^{(i)}) + O((\ln s)^{-\infty}) . \quad (3.21)$$

Passing to Fourier transforms we have

$$\hat{e}(k) = 2g\sqrt{2\pi} \frac{e^{-\frac{|k|}{2}}}{k} J_1(\sqrt{2}gk) \quad (3.22)$$

and, consequently,

$$\gamma(g, s, L) = - \int_{-\infty}^{+\infty} \frac{dk}{4\pi^2} 2g\sqrt{2\pi} \frac{e^{-\frac{|k|}{2}}}{k} J_1(\sqrt{2}gk) \left[ \hat{\sigma}_H(k) + \hat{\sigma}_0^{(s)}(k) + 2\pi \sum_{h=1}^{L-2} \cos ku_h^{(i)} \right] + O((\ln s)^{-\infty}) . \quad (3.23)$$

Comparing (3.23) with (3.13), we see that

$$\gamma(g, s, L) = \frac{1}{\pi} \lim_{k \rightarrow 0} \hat{\sigma}_H(k) + O((\ln s)^{-\infty}) , \quad (3.24)$$

which extends the Kotikov-Lipatov relation [40] to all the sublogarithmic  $O((\ln s)^{-n})$ ,  $n \geq 1$ , contributions and allows to compute the high spin anomalous dimension from the higher than one loop density.

For computational reasons, it is more convenient to use the function

$$S(k) = \frac{\sinh \frac{|k|}{2}}{\pi |k|} \left\{ \hat{\sigma}_H(k) - \pi \frac{e^{-\frac{|k|}{2}}}{\sinh \frac{|k|}{2}} \sum_{h=1}^{L-2} \left[ \cos ku_h^{(i)} - \cos k\bar{u}_h^{(i)} \right] \right\} \Rightarrow \gamma(g, s, L) = 2 \lim_{k \rightarrow 0} S(k) . \quad (3.25)$$

The function (3.25) satisfies the integral equation (for  $k > 0$ )

$$\begin{aligned} S(k) &= \frac{L}{k} [1 - J_0(\sqrt{2}gk)] - g^2 \int_0^{+\infty} \frac{dt}{\pi} e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \cdot \\ &\cdot \left\{ \frac{\pi t}{\sinh \frac{t}{2}} S(t) - 4\pi \ln 2 \delta(t) - \pi(L-2) \frac{1 - e^{\frac{t}{2}}}{\sinh \frac{t}{2}} - 2\pi \frac{1 - e^{-\frac{t}{2}} \cos \frac{ts}{\sqrt{2}}}{\sinh \frac{t}{2}} + \right. \\ &+ \pi \frac{e^{\frac{t}{2}}}{\sinh \frac{t}{2}} \sum_{h=1}^{L-2} \left[ \cos tu_h^{(i)} - 1 \right] \Big\} = 4g^2 \ln s \hat{K}(\sqrt{2}gk, 0) + 4g^2 \int_0^{+\infty} \frac{dt}{e^t - 1} \hat{K}^*(\sqrt{2}gk, \sqrt{2}gt) + \\ &+ \frac{L}{k} [1 - J_0(\sqrt{2}gk)] + 4g^2 \gamma_E \hat{K}(\sqrt{2}gk, 0) + g^2(L-2) \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{1 - e^{\frac{t}{2}}}{\sinh \frac{t}{2}} - \\ &- g^2 \int_0^{+\infty} dt \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{\sum_{h=1}^{L-2} [\cos tu_h^{(i)} - 1]}{\sinh \frac{t}{2}} - g^2 \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{t}{\sinh \frac{t}{2}} S(t) , \end{aligned} \quad (3.26)$$

where  $\hat{K}^*(t, t') = \hat{K}(t, t') - \hat{K}(t, 0)$ . We remark that such equation depends on the position of the holes  $u_h^{(i)}$  (but not on both  $\bar{u}_h^{(i)}$ ,  $u_h^{(i)}$ ) and that the quantity  $u_h^{(i)}$  is also an unknown of the problem and has to be determined by solving equation (2.10),

$$Z(u_h^{(i)}) = \pi(2h + 1 - L), \quad h = 1, \dots, L - 2,$$

which introduces non-linear effects in the equation for the density  $S(k)$ . Equation (3.26) is exact if we neglect - in the high spin limit - terms which are  $O((\ln s)^{-\infty})$  and is the main integral equation of this paper<sup>7</sup>. It will be our starting point in order to investigate the high spin limit of (minimal) anomalous dimension of twist operators in the  $sl(2)$  sector.

We will study two cases. First, we will consider the limit:

$$s \rightarrow \infty, \quad L \text{ fixed}. \quad (3.27)$$

Then, we will focus on the scaling limit (1.7) [11]:

$$s \rightarrow \infty, \quad L \rightarrow \infty, \quad j = \frac{L - 2}{\ln s} \text{ fixed}.$$

The case (3.27) will be reported in Section 4, the case (1.7) in Section 5.

## 4 High spin at fixed twist

In this section we report our results on the fixed twist case (3.27). We naturally make a move from equation (3.26), in which the various contributions to the known ('forcing') term are separated according to their power of  $\ln s$ . As a consequence of the structure of the forcing term, the high spin expansion of the function  $S(k)$  goes on as a series of logarithmic (inverse) powers (1.6):

$$S(k) = \sum_{n=-1}^{\infty} S^{(n)}(k) (\ln s)^{-n} + O((\ln s)^{-\infty}). \quad (4.1)$$

Consequently, the anomalous dimension at high spin follows the same fate:

$$\gamma(g, s, L) = f(g) \ln s + f_{sl}(g, L) + \sum_{n=1}^{\infty} \gamma^{(n)}(g, L) (\ln s)^{-n} + O((\ln s)^{-\infty}).$$

Coming back to  $S(k)$ , the term proportional to  $\ln s$  satisfies the BES linear integral equation,

$$S^{(-1)}(k) = 4g^2 \hat{K}(\sqrt{2}gk, 0) - g^2 \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{t}{\sinh \frac{t}{2}} S^{(-1)}(t), \quad (4.2)$$

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<sup>7</sup>Actually, for twist two there is a simple way to get also the  $O(\frac{\ln s}{s})$  and  $O(\frac{1}{s})$  terms from the first of (3.26): it is sufficient to replace in the argument of the cos function the quantity  $\frac{s}{\sqrt{2}}$ , which represent the leading contribution to the position of the external hole at large  $s$ , with the more accurate estimate [41]  $\frac{s}{\sqrt{2}} (1 + \frac{\gamma+1}{2s} + O(1/s^2))$ . We then get the same result as (29) of [26].

whose solution determines the universal scaling function through  $f(g) = 2S^{(-1)}(0)$ . The BES equation was introduced in [12, 6] and thoroughly studied in the weak [12, 6] and the strong coupling [20, 21, 22, 23] limit. The interested reader can refer to these papers for a detailed study of equation (4.2).

The four subsequent terms - independent of  $s$  - appear in the linear integral equation for the density which determines the virtual scaling function  $f_{sl}(g, L)$ . This equation was written in [25] (equation 4.11), then it was re-obtained in [26] and used there and in our contemporaneous paper [27] (where it appears as equation 3.3). In notations used in this paper it reads ( $k \geq 0$ )

$$\begin{aligned} S^{(0)}(k) = & 4g^2 \int_0^{+\infty} \frac{dt}{e^t - 1} \hat{K}^*(\sqrt{2}gk, \sqrt{2}gt) + \\ & + \frac{L}{k} [1 - J_0(\sqrt{2}gk)] + 4g^2 \gamma_E \hat{K}(\sqrt{2}gk, 0) + g^2 (L - 2) \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{1 - e^{\frac{t}{2}}}{\sinh \frac{t}{2}} - \\ & - g^2 \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{t}{\sinh \frac{t}{2}} S^{(0)}(t), \end{aligned} \quad (4.3)$$

the virtual scaling function  $f_{sl}(g, L)$  being

$$f_{sl}(g, L) = 2S^{(0)}(0). \quad (4.4)$$

As in the case of  $f(g)$ , an explicit expression for  $f_{sl}(g, L)$ , interpolating from weak to strong coupling has not been found yet. What we did is expanding equation (4.3) in a systematic way for small  $g$ , thus getting the weak coupling convergent series for  $f_{sl}(g, L)$  (formula (4.1) of [27]):

$$\begin{aligned} f_{sl}(g, L) = & (\gamma_E - (L - 2) \ln 2) f(g) + 8(2L - 7) \zeta(3) \left( \frac{g}{\sqrt{2}} \right)^4 + \\ & - \frac{8}{3} (\pi^2 \zeta(3) (L - 4) + 3(21L - 62) \zeta(5)) \left( \frac{g}{\sqrt{2}} \right)^6 + \\ & + \frac{8}{15} (\pi^4 \zeta(3) (3L - 13) + 75(46L - 127) \zeta(7) + 5(11L - 32) \pi^2 \zeta(5)) \left( \frac{g}{\sqrt{2}} \right)^8 + \\ & - \left( \frac{128}{945} \pi^6 \zeta(3) (11L - 49) + 8(2695 \zeta(9) L + 16 \zeta(3)^3 L - 7156 \zeta(9) + 56 \zeta(3)^3) + \right. \\ & \left. + \frac{40}{3} (25L - 64) \pi^2 \zeta(7) + \frac{8}{45} (103L - 310) \pi^4 \zeta(5) \right) \left( \frac{g}{\sqrt{2}} \right)^{10} + \\ & + \left( \frac{32}{45} \pi^4 \zeta(7) (295L - 772) + \frac{8}{3} \pi^2 (1519 \zeta(9) L + 24 \zeta(3)^3 L - 3628 \zeta(9) - 88 \zeta(3)^3) + \right. \\ & + 8(33285 \zeta(11) L + 536 \zeta(3)^2 \zeta(5) L - 86082 \zeta(11) - 1728 \zeta(3)^2 \zeta(5)) + \\ & \left. + \frac{8}{945} (2023L - 6266) \pi^6 \zeta(5) + \frac{8(2956L - 13231) \pi^8 \zeta(3)}{14175} \right) \left( \frac{g}{\sqrt{2}} \right)^{12} + \dots \end{aligned} \quad (4.5)$$

In addition, in [27] we performed the strong coupling analysis, by means of analytical and numerical

computations. For the first leading terms (formula (4.12) of [27]) we provided the following outcome:

$$f_{sl}(g, L) = 2\sqrt{2}g \left[ \ln \frac{2\sqrt{2}}{g} - c_1 - \frac{3 \ln 2}{2\sqrt{2}\pi g} \ln \frac{2\sqrt{2}}{g} + \frac{c_0 + (2-L)\pi}{2\sqrt{2}\pi g} - \frac{K}{8\pi^2 g^2} \ln \frac{2\sqrt{2}}{g} + \frac{k_{-1}}{2\sqrt{2}g^2} + O\left(\frac{\ln g}{g^3}\right) \right], \quad (4.6)$$

where  $c_1 = 1$ ,  $c_0 = 6 \ln 2 - \pi$ ,  $K = \beta(2)$  is the Catalan's constant and (see also [26])

$$k_{-1} = \frac{4K - 9(\ln 2)^2}{4\sqrt{2}\pi^2} = -0.0118253 \dots \quad (4.7)$$

Importantly, in this asymptotic expansion we could show that the only trace of the twist comes up in the piece  $c_0 + (2-L)\pi$  and thus cancels out completely, at order  $O(s^0)$ , in the asymptotic (large  $g$ ) expansion of  $\Delta - s = \gamma + L$ . It follows that the constant term (i.e.  $O(g^0)$ ) in  $\Delta - s = \gamma + L$  at order  $O(s^0)$  is  $\frac{6 \ln 2 + \pi}{\pi}$  for any twist: this allowed successful comparisons with string theory results [28], which does not distinguish between null and small values of  $L$ .

An alternative analysis of the strong coupling limit can be found in [26]. This computation was done adapting the method introduced in [22] for  $f(g)$ .

For what concerns  $\gamma^{(n)}(g, L)$ , i.e. the  $O((\ln s)^{-n})$ ,  $n \geq 1$ , contributions to the anomalous dimensions, they are 'driven' by the holes depending parts of (3.26). As a consequence of (3.24), one has  $\gamma^{(n)}(g, L) = 2S^{(n)}(0)$ , where  $S^{(n)}(k)$  satisfies the integral equation ( $k > 0$ )

$$\begin{aligned} S^{(n)}(k) &= -g^2 \int_0^{+\infty} dt \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{\sum_{h=1}^{L-2} [\cos tu_h^{(i)} - 1]}{\sinh \frac{t}{2}} \Big|_{(\ln s)^{-n}} - \\ &- g^2 \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{t}{\sinh \frac{t}{2}} S^{(n)}(t), \end{aligned} \quad (4.8)$$

with the symbol  $|_{(\ln s)^{-n}}$  standing for the component proportional to  $(\ln s)^{-n}$ . The analysis of this case was done in detail in the paper [30]. We report here the main results.

In order to fulfil condition (2.10), the position of the internal holes has to expand in inverse powers of  $\ln s$ :

$$u_h^{(i)} = \sum_{n=1}^{\infty} \alpha_{n,h} (\ln s)^{-n} + O((\ln s)^{-\infty}). \quad (4.9)$$

Introducing the (even) derivatives in zero of the (even) function  $\sigma(u) = Z'(u)$ , developing them in powers of  $\ln s$ ,

$$\frac{d^{2q}}{du^{2q}} \sigma(u=0) = \sum_{n=-1}^{\infty} \sigma_{2q}^{(n)} (\ln s)^{-n}, \quad (4.10)$$

and imposing the condition (2.10) for the holes we eventually get the following recursive equation for the unknowns  $\alpha_{n,h}$ ,

$$\begin{aligned} \alpha_{p+1,h} &= - \sum_{r=1}^p \frac{\sigma_r^{(-1)}}{\sigma_0^{(-1)}} \sum_{\{j_1, \dots, j_{p-r+1}\}} \prod_{m=1}^{p-r+1} \frac{(\alpha_{m,h})^{j_m}}{j_m!} - \sum_{l=0}^{p-1} \sum_{r=1}^{p-l} \frac{\sigma_{r-1}^{(l)}}{\sigma_0^{(-1)}} \sum_{\{j_1, \dots, j_{p-r-l+1}\}} \prod_{m=1}^{p-r-l+1} \frac{(\alpha_{m,h})^{j_m}}{j_m!}, \quad p \geq 1 \\ \alpha_{1,h} &= \frac{\pi(2h-1+L)}{\sigma_0^{(-1)}}, \quad p = 0, \end{aligned} \quad (4.11)$$

where the  $j_m$  contained in the second term of the right hand side are constrained by the conditions  $\sum_{m=1}^{p-r+1} j_m = r+1$ ,  $\sum_{m=1}^{p-r+1} m j_m = p+1$  and the ones in the third term by  $\sum_{m=1}^{p-r-l+1} j_m = r$ ,  $\sum_{m=1}^{p-r-l+1} m j_m = p-l$ .

The next step is the Neumann expansion for the functions  $S^{(n)}(k)$  (in the domain  $k > 0$ ):

$$S^{(n)}(k) = \sum_{p=1}^{\infty} S_p^{(n)}(g) \frac{J_p(\sqrt{2}gk)}{k} \Rightarrow \gamma^{(n)}(g, L) = \sqrt{2}g S_1^{(n)}(g). \quad (4.12)$$

Straightforward but lengthy calculations, originating from equation (3.26), lead to the conclusion that the Neumann modes  $S_p^{(n)}(g)$  satisfy the system<sup>8</sup>

$$\begin{aligned} S_{2p-1}^{(n)}(g) &= -(2p-1) \int_0^{+\infty} \frac{dt}{t} \frac{\mathcal{P}_n(g, t) J_{2p-1}(\sqrt{2}gt)}{\sinh \frac{t}{2}} - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1, m}(g) S_m^{(n)}(g), \\ S_{2p}^{(n)}(g) &= -2p \int_0^{+\infty} \frac{dt}{t} \frac{\mathcal{P}_n(g, t) J_{2p}(\sqrt{2}gt)}{\sinh \frac{t}{2}} - 4p \sum_{m=1}^{\infty} Z_{2p, m}(g) (-1)^m S_m^{(n)}(g). \end{aligned} \quad (4.14)$$

In (4.14)  $\mathcal{P}_n(g, t)$  appears as a coefficient in the high spin expansion

$$P(s, g, t) = \sum_{n=1}^{\infty} \mathcal{P}_n(g, t) (\ln s)^{-n} \quad (4.15)$$

of the internal holes-depending function

$$P(s, g, t) = \sum_{h=1}^{L-2} \left[ \cos tu_h^{(i)} - 1 \right], \quad (4.16)$$

appearing in (4.8). As a consequence of (4.11),  $\mathcal{P}_n(g, t)$  depends on the various coefficients  $\alpha_{m, h}$  of (4.9) as

$$\mathcal{P}_n(g, t) = \sum_{r=1}^n t^r \cos \frac{\pi r}{2} \sum_{\{j_1, \dots, j_{n-r+1}\}} \frac{\sum_{h=1}^{L-2} \prod_{m=1}^{n-r+1} (\alpha_{m, h})^{j_m}}{\prod_{m=1}^{n-r+1} j_m!}, \quad \sum_{m=1}^{n-r+1} j_m = r, \quad \sum_{m=1}^{n-r+1} m j_m = n. \quad (4.17)$$

In order to study the system (4.14), it is useful to introduce the “reduced coefficients”  $\tilde{S}_p^{(k)}$ , defined in equations (4.23, 4.24) of [34] as solutions of the reduced systems

$$\tilde{S}_{2p}^{(k)}(g) = \mathbb{I}_{2p}^{(k)}(g) - 4p \sum_{m=1}^{\infty} Z_{2p, m}(g) (-1)^m \tilde{S}_m^{(k)}(g), \quad (4.18)$$

$$\tilde{S}_{2p-1}^{(k)}(g) = \mathbb{I}_{2p-1}^{(k)}(g) - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1, m}(g) \tilde{S}_m^{(k)}(g),$$

---

<sup>8</sup>We use the notation:

$$Z_{n, m}(g) = \int_0^{+\infty} \frac{dt}{t} \frac{J_n(\sqrt{2}gt) J_m(\sqrt{2}gt)}{e^t - 1}. \quad (4.13)$$

with the explicit (i.e. not depending on the densities  $\sigma_{2q}^{(n)}$ ) forcing terms,

$$\mathbb{I}_r^{(k)} = r \int_0^{+\infty} \frac{dh}{2\pi} h^{2k-1} \frac{J_r(\sqrt{2}gh)}{\sinh \frac{h}{2}}. \quad (4.19)$$

Indeed, solutions to the systems (4.14) are linear combinations of the various  $\tilde{S}_r^{(k)}(g)$  with coefficients depending on  $\alpha_{n,h}$ . In particular, for what concerns  $\gamma^{(n)}(g)$ , its expression in terms of  $\tilde{S}_1^{(k)}$  and  $\alpha_{m,h}$  reads

$$\frac{\gamma^{(n)}(g)}{\sqrt{2}g} = -2\pi \sum_{r=1}^n \tilde{S}_1^{(r/2)} \cos \frac{\pi r}{2} \sum_{\{j_1, \dots, j_{n-r+1}\}} \frac{\sum_{h=1}^{L-2} \prod_{m=1}^{n-r+1} (\alpha_{m,h})^{j_m}}{\prod_{m=1}^{n-r+1} j_m!}, \quad \sum_{m=1}^{n-r+1} j_m = r, \quad \sum_{m=1}^{n-r+1} m j_m = n. \quad (4.20)$$

The various  $\alpha_{n,h}$  are written in terms of the densities with the help of (4.11), leaving eventually  $\gamma^{(n)}(g)$  as depending on  $\tilde{S}_1^{(k)}$  and  $\sigma_{2q}^{(r)}$ , with  $r \leq n-3$ . This last property is very important, since it makes possible to build up a recursive calculation scheme for the  $\gamma^{(n)}(g)$ , opening the way to push the computation up to the desired order in  $\ln s$ .

As an example in paper [30] we gave the following exact results for the first  $\gamma^{(n)}(g)$ :

$$\gamma^{(1)}(g, L) = 0, \quad (4.21)$$

$$\gamma^{(2)}(g, L) = \sqrt{2}g \frac{\pi^3}{3(\sigma_0^{(-1)})^2} (L-3)(L-2)(L-1) \tilde{S}_1^{(1)}(g), \quad (4.22)$$

$$\gamma^{(3)}(g, L) = -2\sqrt{2}g \frac{\pi^3 \sigma_0^{(0)}}{3(\sigma_0^{(-1)})^3} (L-3)(L-2)(L-1) \tilde{S}_1^{(1)}(g), \quad (4.23)$$

$$\begin{aligned} \gamma^{(4)}(g, L) = & \sqrt{2}g 2\pi \left\{ \left[ -\frac{\pi^2}{3(\sigma_0^{(-1)})^2} \left( \frac{\sigma_0^{(1)}}{\sigma_0^{(-1)}} - \frac{3}{2} \frac{(\sigma_0^{(0)})^2}{(\sigma_0^{(-1)})^2} \right) (L-3)(L-2)(L-1) - \right. \right. \\ & - \frac{\pi^4 \sigma_2^{(-1)}}{90(\sigma_0^{(-1)})^5} (L-3)(L-2)(L-1)(5+3L(L-4)) \left. \right] \tilde{S}_1^{(1)} - \\ & \left. - \frac{\pi^4}{360(\sigma_0^{(-1)})^4} (L-3)(L-2)(L-1)(5+3L(L-4)) \tilde{S}_1^{(2)} \right\}. \end{aligned} \quad (4.24)$$

Due to the aforementioned recursive properties, such expressions can be explicitly computed in the weak and in the strong coupling limit. Weak coupling expansions are presented in Appendix A of [30]. The strong coupling leading term is given in Section 3 of that paper.

## 5 High spin and large twist

We now want to study the anomalous dimension in the limit (1.7):

$$s \rightarrow \infty, \quad L \rightarrow \infty, \quad j = \frac{L-2}{\ln s} \quad \text{fixed}.$$

Since the number of internal holes becomes infinite, we need to treat the sum over them contained in (3.26) by means of the technique discussed in Section 2. Before doing that, we introduce two points  $\pm c$  which separate the internal holes and the Bethe roots: since the number of internal holes equals  $L - 2$ , the 'separator'  $c$  has to satisfy the following relation:

$$Z(c) = \frac{1}{2} \int_{-c}^c dv \sigma(v) = -\pi j \ln s. \quad (5.1)$$

Now we can specialise formula (2.19) to the sum

$$\sum_{h=1}^{L-2} \left[ \cos t u_h^{(i)} - 1 \right], \quad (5.2)$$

over the internal holes  $u_h^{(i)} \in [-c, c]$ . Remember that in  $[-c, c]$  no Bethe roots are present: therefore the right hand side of (2.19) is zero. We get

$$\begin{aligned} \sum_{h=1}^{L-2} \left[ \cos t u_h^{(i)} - 1 \right] &= - \int_{-c}^c \frac{dv}{2\pi} (\cos tv - 1) \sigma(v) - \frac{\pi t \sin tc}{6 \sigma(c)} - \\ &- \frac{7\pi^3 t^3 \sigma(c) \sin tc + 3t^2 \sigma_1(c) \cos tc - 3t \frac{(\sigma_1(c))^2}{\sigma(c)} \sin tc + t \sigma_2(c) \sin tc}{360 (\sigma(c))^4} + O\left(\frac{j^n}{(\ln s)^5}\right) = \\ &= -2 \int_{-\infty}^{+\infty} \frac{dk}{4\pi^2} \hat{\sigma}(k) \left[ \frac{\sin(t+k)c}{t+k} - \frac{\sin kc}{k} \right] - \frac{\pi t \sin tc}{6 \sigma(c)} - \\ &- \frac{7\pi^3 t^3 \sigma(c) \sin tc + 3t^2 \sigma_1(c) \cos tc + t \sigma_2(c) \sin tc}{360 (\sigma(c))^4} + O\left(\frac{j^3}{(\ln s)^3}\right), \end{aligned} \quad (5.3)$$

where  $\sigma_m(c)$  denotes the  $m$ -th derivative of the density  $\sigma(v)$  in  $v = c$ . This expression has to be inserted in (3.26) and worked out together with condition (5.1): we get in general a non-linear integral equation for the quantity  $S(k)$ , related to the Fourier transform of the density of Bethe roots and internal holes  $\hat{\sigma}(k)$  through (3.25).

This joined analysis of (3.26, 5.1) gets simplified in the limit (1.7). Indeed, it follows from the structure of (3.26) that, in the case of limit (1.7), the function  $S(k)$  expands as [24, 35, 30]

$$S(k) = \sum_{r=-1}^{\infty} (\ln s)^{-r} \sum_{n=0}^{\infty} S^{(r,n)}(k) j^n + O((\ln s)^{-\infty}) \quad (5.4)$$

and, correspondingly, the anomalous dimension behaves as (1.8):

$$\gamma(g, s, L) = \ln s \sum_{n=0}^{\infty} f_n(g) j^n + \sum_{r=0}^{\infty} (\ln s)^{-r} \sum_{n=0}^{\infty} f_n^{(r)}(g) j^n + O((\ln s)^{-\infty}).$$

The functions  $f_n(g)$ ,  $f_n^{(r)}(g)$  are called generalised scaling functions. In particular  $f_0(g)$  coincides with the universal scaling function  $f(g)$ .

Similarly, the separator between internal holes and Bethe roots,  $c$ , enjoys the following scaling in the limit (1.7),

$$c = \sum_{r=0}^{\infty} (\ln s)^{-r} \sum_{n=1}^{\infty} c^{(r,n)} j^n. \quad (5.5)$$

The constants  $c^{(r,n)}$  are connected to the various components  $\sigma_{2q}^{(r,n)}$  in which the density and its derivatives in zero expand in the limit (1.7),

$$\frac{d^{2q}}{du^{2q}} \sigma(u=0) = \sum_{r=-1}^{\infty} \sum_{n=0}^{\infty} \sigma_{2q}^{(r,n)} (\ln s)^{-r} j^n, \quad (5.6)$$

by means of (5.1):

$$\frac{1}{2} \int_{-c}^c dv \sigma(v) = -\pi j \ln s.$$

For instance, using (5.5, 5.6) in (5.1), we get, for the first  $c^{(r,n)}$ :

$$\begin{aligned} c^{(0,1)} &= -\frac{\pi}{\sigma^{(-1,0)}}, & c^{(0,2)} &= \pi \frac{\sigma^{(-1,1)}}{[\sigma^{(-1,0)}]^2}, & c^{(0,3)} &= \frac{\pi^3}{6} \frac{\sigma_2^{(-1,0)}}{[\sigma^{(-1,0)}]^4} - \pi \frac{[\sigma^{(-1,1)}]^2}{[\sigma^{(-1,0)}]^3}, \\ c^{(1,1)} &= \pi \frac{\sigma^{(0,0)}}{[\sigma^{(-1,0)}]^2}, & c^{(1,2)} &= -2\pi \frac{\sigma^{(0,0)} \sigma^{(-1,1)}}{[\sigma^{(-1,0)}]^3}, \\ c^{(1,3)} &= 3\pi \frac{\sigma^{(0,0)} [\sigma^{(-1,1)}]^2}{[\sigma^{(-1,0)}]^4} - \frac{2}{3} \pi^3 \frac{\sigma^{(0,0)} \sigma_2^{(-1,0)}}{[\sigma^{(-1,0)}]^5} + \frac{\pi^3}{6} \frac{\sigma_2^{(0,0)}}{[\sigma^{(-1,0)}]^4}, \\ c^{(2,1)} &= -\pi \frac{[\sigma^{(0,0)}]^2}{[\sigma^{(-1,0)}]^3}. \end{aligned} \quad (5.7)$$

Therefore, once we take the limit (1.7) and use condition (5.1), we easily get that equation (3.26) splits in a set of equations, one for every function  $S^{(r,n)}(k)$ . Remember that  $S^{(-1,0)}(k)$  coincides with the BES density (function  $S^{(-1)}(k)$  of last section): therefore, it satisfies the BES linear equation (4.2), i.e.

$$S^{(-1,0)}(k) = 4g^2 \hat{K}(\sqrt{2}gk, 0) - g^2 \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{t}{\sinh \frac{t}{2}} S^{(-1,0)}(t).$$

Another particular case is the function  $S^{(-1,1)}(k)$ : this satisfies the linear equation

$$\begin{aligned} S^{(-1,1)}(k) &= \frac{1}{k} [1 - J_0(\sqrt{2}gk)] + g^2 \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{1 - e^{\frac{t}{2}}}{\sinh \frac{t}{2}} - \\ &- g^2 \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{t}{\sinh \frac{t}{2}} S^{(-1,1)}(t), \end{aligned} \quad (5.8)$$

and determines the generalised scaling function  $f_1(g) = 2S^{(-1,1)}(0)$ . Equation (5.8) was studied in paper [31].

The last case that has to be treated separately is  $r = j = 0$ . However, in this case, the density is obtained from  $S^{(0)}(k)$  ( $(\ln s)^0$  contribution at fixed twist, see last section) and  $S^{(-1,1)}(k)$ , by means of the equality

$$S^{(0)}(k) = (L - 2)S^{(-1,1)}(k) + S^{(0,0)}(k). \quad (5.9)$$

In the three cases we have discussed up to now, the dynamics of the internal holes is not relevant: for the required approximations the internal holes can be supposed all lying at the origin, i.e. the sum (5.2) can be considered as vanishing.

The particular form of the expansion (5.5) with coefficients given by (5.7) is however of fundamental importance for all the remaining cases. They can be studied together, by means of the integral equation

$$S^{(r,n)}(k) = -g^2 \int_0^{+\infty} dt \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{\sum_{h=1}^{L-2} [\cos tu_h^{(i)} - 1]}{\sinh \frac{t}{2}} \Big|_{\frac{j^n}{(\ln s)^r}} -$$

$$- g^2 \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{t}{\sinh \frac{t}{2}} S^{(r,n)}(t), \quad (5.10)$$

in which the sum over the internal holes is evaluated through (5.3) and the 'separator'  $c$  is expanded as in (5.5), with coefficients depending on the quantities  $\sigma_{2q}^{(r,n)}$  and determined by the use of (5.1).

From the methodological point of view, we need to distinguish two cases. The first one covers the values  $r = -1, n \geq 2$  and  $r = 0, n \geq 1$ . In this case only the first term in the right hand side of (5.3), which is linear in the density, is relevant. Therefore in the final equation non-linearity comes only from the non-linear dependence of  $c$  on the density and their derivatives in zero:

$$S^{(r,n)}(k) = g^2 \int_0^{+\infty} dt \frac{\hat{K}(\sqrt{2}gk, \sqrt{2}gt)}{\sinh \frac{t}{2}} \int_{-c}^c \frac{dv}{2\pi} (\cos tv - 1) \sigma(v) \Big|_{\frac{j^n}{(\ln s)^r}} -$$

$$- g^2 \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{t}{\sinh \frac{t}{2}} S^{(r,n)}(t). \quad (5.11)$$

For instance, using (5.7) we get for the first cases:

$$\begin{aligned} & \int_{-c}^c \frac{dv}{2\pi} (\cos tv - 1) \sigma(v) \Big|_{\ln s \cdot j^2} = 0, \\ & \int_{-c}^c \frac{dv}{2\pi} (\cos tv - 1) \sigma(v) \Big|_{\ln s \cdot j^3} = \frac{1}{6} \pi^2 \frac{t^2}{[\sigma^{(-1,0)}]^2}, \quad \int_{-c}^c \frac{dv}{2\pi} (\cos tv - 1) \sigma(v) \Big|_{\ln s \cdot j^4} = -\frac{1}{3} \pi^2 \frac{t^2 \sigma^{(-1,1)}}{[\sigma^{(-1,0)}]^3}, \\ & \int_{-c}^c \frac{dv}{2\pi} (\cos tv - 1) \sigma(v) \Big|_{\ln s \cdot j^5} = -\frac{\pi^4 t^4}{120 [\sigma^{(-1,0)}]^4} + \frac{\pi^2 [\sigma^{(-1,1)}]^2 t^2}{2 [\sigma^{(-1,0)}]^4} - \frac{\pi^4 \sigma_2^{(-1,0)} t^2}{30 [\sigma^{(-1,0)}]^5}, \\ & \int_{-c}^c \frac{dv}{2\pi} (\cos tv - 1) \sigma(v) \Big|_{\frac{j^r}{(\ln s)^0}} = 0, \quad r = 1, 2, \quad \int_{-c}^c \frac{dv}{2\pi} (\cos tv - 1) \sigma(v) \Big|_{\frac{j^3}{(\ln s)^0}} = -\frac{1}{3} \pi^2 \frac{\sigma^{(0,0)}}{[\sigma^{(-1,0)}]^3} \tilde{S}_1^{(1)}(g), \\ & \int_{-c}^c \frac{dv}{2\pi} (\cos tv - 1) \sigma(v) \Big|_{\frac{j^4}{(\ln s)^0}} = \pi^2 \frac{\sigma^{(0,0)} \sigma^{(-1,1)}}{[\sigma^{(-1,0)}]^4} \tilde{S}_1^{(1)}(g), \\ & \int_{-c}^c \frac{dv}{2\pi} (\cos tv - 1) \sigma(v) \Big|_{\frac{j^5}{(\ln s)^0}} = -\left[ \frac{1}{6} \left( \frac{\pi^4}{5} \frac{\sigma_2^{(0,0)}}{[\sigma^{(-1,0)}]^5} + 12 \pi^2 \frac{\sigma^{(0,0)} [\sigma^{(-1,1)}]^2}{[\sigma^{(-1,0)}]^5} - \pi^4 \frac{\sigma^{(0,0)} \sigma_2^{(-1,0)}}{[\sigma^{(-1,0)}]^6} \right) \tilde{S}_1^{(1)}(g) - \right. \\ & \left. - \frac{\pi^4}{30} \frac{\sigma^{(0,0)}}{[\sigma^{(-1,0)}]^5} \tilde{S}_1^{(2)}(g) \right]. \end{aligned} \quad (5.12)$$

In the remaining cases, i.e.  $r \geq 1$ , the evaluation of the sum over the internal holes (5.2) involves also terms explicitly non-linear in the density. Thanks to formulæ (2.19, 5.3), however, everything is under control and, for instance, for the first values of  $r, n$  we get

$$\begin{aligned}
\sum_{h=1}^{L-2} \left[ \cos tu_h^{(i)} - 1 \right] \Big|_{\frac{j^0}{(\ln s)^r}} &= 0, \quad r = 1, 2, 3, 4, \\
\sum_{h=1}^{L-2} \left[ \cos tu_h^{(i)} - 1 \right] \Big|_{\frac{j}{\ln s}} &= \frac{\pi^2}{6} \frac{t^2}{(\sigma^{(-1,0)})^2}, \quad \sum_{h=1}^{L-2} \left[ \cos tu_h^{(i)} - 1 \right] \Big|_{\frac{j^2}{\ln s}} = -\frac{\pi^2}{3} t^2 \frac{\sigma^{(-1,1)}}{(\sigma^{(-1,0)})^3}, \\
\sum_{h=1}^{L-2} \left[ \cos tu_h^{(i)} - 1 \right] \Big|_{\frac{j^3}{\ln s}} &= -\frac{\pi^2}{2} t^2 \frac{(\sigma^{(0,0)})^2}{(\sigma^{(-1,0)})^4} - \frac{\pi^2}{6} \frac{t^2}{(\sigma^{(-1,0)})^4} \left[ \frac{2}{3} \pi^2 \frac{\sigma_2^{(-1,0)}}{\sigma^{(-1,0)}} - 3(\sigma^{(-1,1)})^2 + \frac{t^2}{6} \pi^2 \right], \\
\sum_{h=1}^{L-2} \left[ \cos tu_h^{(i)} - 1 \right] \Big|_{\frac{j}{(\ln s)^2}} &= -\frac{\pi^2}{3} t^2 \frac{\sigma^{(0,0)}}{(\sigma^{(-1,0)})^3}, \quad \sum_{h=1}^{L-2} \left[ \cos tu_h^{(i)} - 1 \right] \Big|_{\frac{j^2}{(\ln s)^2}} = \pi^2 t^2 \frac{\sigma^{(0,0)} \sigma^{(-1,1)}}{(\sigma^{(-1,0)})^4}, \\
\sum_{h=1}^{L-2} \left[ \cos tu_h^{(i)} - 1 \right] \Big|_{\frac{j}{(\ln s)^3}} &= \frac{7\pi^4}{360} t^4 \frac{1}{(\sigma^{(-1,0)})^4} + \frac{7\pi^4}{90} t^2 \frac{\sigma_2^{(-1,0)}}{(\sigma^{(-1,0)})^5} + \frac{\pi^2}{2} t^2 \frac{(\sigma^{(0,0)})^2}{(\sigma^{(-1,0)})^4}. \tag{5.13}
\end{aligned}$$

In both cases, next steps are the usual ones and details can be found in [34, 35, 30]. First, we perform a Neumann expansion for the even functions  $S^{(r,n)}(k)$ , in the domain  $k \geq 0$ :

$$S^{(r,n)}(k) = \sum_{p=1}^{\infty} S_p^{(r,n)}(g) \frac{J_p(\sqrt{2}gk)}{k}. \tag{5.14}$$

This implies that the generalised scaling functions are expressed as

$$f_n^{(r)}(g) = \sqrt{2}g S_1^{(r,n)}(g). \tag{5.15}$$

The Neumann modes  $S_p^{(r,n)}(g)$  satisfy linear systems and are linear combinations of the "reduced" coefficients  $\tilde{S}_p^{(k)}$  which are solutions of the systems (4.18). The coefficients driving such linear combinations depend on the densities and their derivatives in zero,  $\sigma_{2q}^{(r',n')}$ , with  $r' \leq r, n' \leq n-1$ . This property allows to find, step by step, exact expressions for the generalised scaling functions  $f_n^{(r)}(g)$  in terms of

$\sigma_{2q}^{(r',n')}$  and  $\tilde{S}_1^{(k)}$ . For the first of them we get the following results:

$$\begin{aligned}
\frac{f_3(g)}{\sqrt{2}g} &= \frac{1}{3}\pi^3 \frac{\tilde{S}_1^{(1)}(g)}{[\sigma^{(-1,0)}]^2}, \quad \frac{f_4(g)}{\sqrt{2}g} = -\frac{2}{3}\pi^3 \frac{\tilde{S}_1^{(1)}(g)\sigma^{(-1,1)}}{[\sigma^{(-1,0)}]^3}, \\
\frac{f_5(g)}{\sqrt{2}g} &= -\frac{\pi^5 \tilde{S}_1^{(2)}(g)}{60[\sigma^{(-1,0)}]^4} + \frac{\pi^3 [\sigma^{(-1,1)}]^2 \tilde{S}_1^{(1)}(g)}{[\sigma^{(-1,0)}]^4} - \frac{\pi^5 \sigma_2^{(-1,0)} \tilde{S}_1^{(1)}(g)}{15[\sigma^{(-1,0)}]^5}, \\
\frac{f_3^{(0)}(g)}{\sqrt{2}g} &= -\frac{2}{3}\pi^3 \frac{\sigma^{(0,0)}}{[\sigma^{(-1,0)}]^3} \tilde{S}_1^{(1)}(g), \quad \frac{f_4^{(0)}(g)}{\sqrt{2}g} = 2\pi^3 \frac{\sigma^{(0,0)}\sigma^{(-1,1)}}{[\sigma^{(-1,0)}]^4} \tilde{S}_1^{(1)}(g), \\
\frac{f_5^{(0)}(g)}{\sqrt{2}g} &= \left[ -\frac{1}{3} \left( \frac{\pi^5}{5} \frac{\sigma_2^{(0,0)}}{[\sigma^{(-1,0)}]^5} + 12\pi^3 \frac{\sigma^{(0,0)}[\sigma^{(-1,1)}]^2}{[\sigma^{(-1,0)}]^5} - \pi^5 \frac{\sigma^{(0,0)}\sigma_2^{(-1,0)}}{[\sigma^{(-1,0)}]^6} \right) \tilde{S}_1^{(1)}(g) + \right. \\
&\quad \left. + \frac{\pi^5}{15} \frac{\sigma^{(0,0)}}{[\sigma^{(-1,0)}]^5} \tilde{S}_1^{(2)}(g) \right], \\
f_0^{(r)} &= 0, \quad r = 1, 2, 3, 4; \quad \frac{f_1^{(1)}}{\sqrt{2}g} = -\frac{\pi^2}{3} \frac{\tilde{S}_1^{(1)}(g)}{(\sigma^{(-1,0)})^2}, \quad \frac{f_2^{(1)}}{\sqrt{2}g} = \frac{2\pi^3}{3} \frac{\sigma^{(-1,1)}}{(\sigma^{(-1,0)})^3} \tilde{S}_1^{(1)}(g), \\
\frac{f_3^{(1)}}{\sqrt{2}g} &= \frac{\pi^3}{(\sigma^{(-1,0)})^4} \left[ \left( \frac{2}{9} \pi^2 \frac{\sigma_2^{(-1,0)}}{\sigma^{(-1,0)}} - (\sigma^{(-1,1)})^2 + (\sigma^{(0,0)})^2 \right) \tilde{S}_1^{(1)}(g) + \frac{\pi^2}{18} \tilde{S}_1^{(2)}(g) \right], \\
\frac{f_1^{(2)}}{\sqrt{2}g} &= \frac{2\pi^3}{3} \frac{\sigma^{(0,0)}}{(\sigma^{(-1,0)})^3} \tilde{S}_1^{(1)}(g), \quad \frac{f_2^{(2)}}{\sqrt{2}g} = -2\pi^3 \frac{\sigma^{(0,0)}\sigma^{(-1,1)}}{(\sigma^{(-1,0)})^4} \tilde{S}_1^{(1)}(g), \\
\frac{f_1^{(3)}}{\sqrt{2}g} &= -\frac{\pi^3}{(\sigma^{(-1,0)})^4} \left[ \frac{7\pi^2}{180} \tilde{S}_1^{(2)}(g) + \left( \frac{7\pi^2}{45} \frac{\sigma_2^{(-1,0)}}{\sigma^{(-1,0)}} + (\sigma^{(0,0)})^2 \right) \tilde{S}_1^{(1)}(g) \right].
\end{aligned}$$

Explicit expressions in the weak and strong coupling limit for the various  $f_n^{(r)}(g)$  can be given without much ado, because of the iterative structure of the relations which determine them. They can be found in [34, 35, 30].

In particular, strong coupling limit of the anomalous dimension is of interest since it can be checked against string theory calculations. For what concerns the function

$$f(g, j) = \sum_{n=0}^{\infty} f_n(g) j^n, \quad (5.16)$$

performing such a check is not a difficult task, after results by Alday and Maldacena [14]. Introducing (at large  $g$ ) the quantity

$$m(g) = \frac{2^{\frac{5}{8}} \pi}{\Gamma(\frac{5}{4})} g^{\frac{1}{4}} e^{-\frac{\pi g}{\sqrt{2}}} \left[ 1 + O\left(\frac{1}{g}\right) \right], \quad (5.17)$$

in [14] it was proved that in the limit (1.7), when  $g \rightarrow \infty$ ,  $j \ll g$ , with  $j/m(g)$  fixed, the quantity  $f(g, j) + j$  has to coincide with the energy density of the ground state of the  $O(6)$  non-linear sigma model with mass gap  $m(g)$ . When  $j/m(g) \ll 1$  we are in the nonperturbative regime of the  $O(6)$  non-linear sigma model. In this case the energy density can be computed by using Bethe Ansatz related techniques. This computation has been systematically performed in [42]. In order to have agreement

between our calculations for  $f(g, j)$  and computations of [42] for the  $O(6)$  non-linear sigma model, we must have that the quantities  $\Omega_n(g)$  computed in that paper have to be related to  $f_n(g)$  by the relation  $f_n(g) = 2^{n-1}\Omega_n(g)$ . Our results [27] for the strong coupling limit of  $f_3(g), f_4(g), f_5(g)$ <sup>9</sup> are

$$f_3(g) = \frac{\pi^2}{24m(g)} + O\left(e^{-\frac{\pi g}{\sqrt{2}}}\right), \quad (5.18)$$

$$f_4(g) = -\frac{\pi^2}{12[m(g)]^2}\mathcal{S}_1 + O(1), \quad (5.19)$$

$$f_5(g) = -\frac{\pi^4}{640[m(g)]^3} + \frac{\pi^2}{8[m(g)]^3}[\mathcal{S}_1]^2 + O\left(e^{\frac{\pi g}{\sqrt{2}}}\right), \quad (5.20)$$

where we used the compact notations

$$\mathcal{S}_{2s+1} = \frac{1}{\pi^{2s+1}} \sum_{n=0}^{\infty} (-1)^n \left[ \frac{1}{(n + \frac{1}{2})^{2s+1}} + \frac{1}{(n+1)^{2s+1}} \right], \quad (5.21)$$

and agree with corresponding formulæ contained in [42].

## 6 Conclusions

In this paper we have discussed our activity on the study of minimal anomalous dimension of twist operators in the  $sl(2)$  sector of  $\mathcal{N} = 4$  SYM at high spin. We preferred to give a general description of the methods we used: the reader can refer to the original papers for details and for a complete list of our results.

The main tool we used is integral equation (3.26) for the density of Bethe roots. This equation coincides with the exact non-linear integral equation equivalent to the ABA equations of the  $sl(2)$  sector if we neglect terms going to zero faster than any inverse power of the logarithm of the spin. Equation (3.26) is linear, apart from the non-linear dependence of the internal holes positions  $u_h^{(i)}$  on the counting function  $Z(u)$ . These non-linear effects are taken into account by inverting relation (2.10): in the high spin limit such inversion is feasible because of the recursive properties of the equations (4.11) determining the various  $u_h^{(i)}$ . This allows to give explicit expressions (4.20) for the coefficients of the high spin expansion of the anomalous dimension.

When the twist goes to infinity, the internal holes are described by their density and contribute to integral equation (3.26) introducing explicitly non-linear terms. These non-linear terms are evaluated (see Section 5) by using properties and techniques of the non-linear integral equation, which are discussed in Section 2. Their contribution to main equation (3.26) is obtained by applying formula (2.19) and is reported in equations (5.12, 5.13). In this case also, recursive properties of the equations determining the high spin limit of the density are crucial in order to obtain explicit expressions for the anomalous dimension.

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<sup>9</sup>In [27] results for the strong coupling limit of  $f_n(g)$  and checks with  $O(6)$  non-linear sigma model results were performed up to  $n = 8$ .

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